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1982 J. Phys. A: Math. Gen. 15 3539

(http://iopscience.iop.org/0305-4470/15/11/031)

J. Phys. A: Math. Gen. 15 (1982) 3539–3542. Printed in Great Britain

Conformally flat solution of a static dust sphere in Einstein–Cartan theory

M L Bedran and M M Som

Universidade Federal do Rio de Janeiro, Instituto de Física, Ilha do Fundão, Rio de Janeiro, Brazil

Received 27 April 1982

Abstract. A unique class of solutions of the classical spin fluid with zero pressure is obtained. The spin density vanishes at the boundary of the fluid sphere and the $g_{\mu\nu}$ and their derivatives satisfy continuity conditions with the exterior metric.

1. Introduction

In an interesting review work, Hehl *et al* (1976) have discussed a generalisation of Einstein's gravitational theory in which the spin of matter, as well as its mass, plays a dynamical role. The spin of matter couples to a non-Riemannian structure in space–time, Cartan's torsion tensor. It is pointed out there that one can always rewrite the Einstein–Cartan equations such that the torsion effects are included in the energy–momentum tensor of matter. In this formalism the classical spin density, which is related algebraically to the torsion tensor, behaves as a repulsive field in a manifest way. Unlike the electrostatic field, it contributes negatively to the energy–momentum tensor.

Following this approach of Hehl *et al*, Prasanna (1975) has constructed some solutions of the Einstein–Cartan equations with reference to the static spherically and cylindrically symmetric perfect fluid. In a recent work, Som *et al* (1982) have shown that one can use the known solutions of the Einstein equations to obtain the solutions of the Einstein–Cartan equations for a static perfect fluid, irrespective of any symmetry. Som and Bedran (1981) have constructed a special class of static solutions of the Einstein–Cartan equations with reference to a fluid with zero pressure.

In this paper we have studied the equilibrium distribution of a dust sphere with the spins of the constituent particles aligned along the radial directions. The number of independent equations is three, with four unknowns. To have the system well determined one imposes in general a coordinate condition which has no invariant significance. In the present work the solution is obtained by imposing a coordinate-independent condition on the system.

2. The Einstein–Cartan equations

The field equations in the Einstein–Cartan theory are given by

$$R^i{}_j - \tfrac{1}{2}\delta^i{}_j R = -8\pi t^i{}_j, \qquad (2.1)$$

0305-4470/82/113539+04$02.00

$$Q^i{}_{jk} - \delta^i{}_j Q^l{}_{lk} - \delta^i{}_k Q^l{}_{jl} = -8\pi S^i{}_{jk}, \tag{2.2}$$

where $t^i{}_j$, $Q^i{}_{jk}$ and $S^i{}_{jk}$ are the canonical energy–momentum tensor of matter, the torsion tensor and the spin density tensor respectively.

For the classical description, the spin density tensor is related to the intrinsic angular momentum tensor S_{ij} by

$$S^i{}_{jk} = V^i S_{jk} \qquad \text{with } V^i S_{ij} = 0 \tag{2.3}$$

where V^i is the four-velocity vector field, having the norm given by

$$V^i V_i = 1. \tag{2.4}$$

The canonical energy–momentum tensor in (2.1) is defined as

$$t^i{}_j = T^i{}_j + \tfrac{1}{2} g^{il} \nabla_k S^k{}_{jl} \tag{2.5}$$

where $T^i{}_j$ is the usual energy–momentum tensor of matter and ∇_k the usual covariant derivative. From the Bianchi identities we have the conservation laws

$$\nabla_i t^i{}_j = S^k{}_{jl} R^l{}_k + \tfrac{1}{2} S^k{}_{mn} R^{mn}{}_{kj} + S^k{}_{mk}(R^m{}_j - \tfrac{1}{2}\delta^m{}_j R). \tag{2.6}$$

3. Static spherically symmetric dust

For a static spherically symmetric dust we consider the line element in the isotropic form

$$ds^2 = -e^{\mu}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + e^{\nu}\, dt^2 \tag{3.1}$$

where μ and ν are functions of r alone. We now consider the simplest case where the spins of the constituent particles of the fluid are aligned in the radial directions. The only non-zero component of S_{ij} is then $S_{23} = e^{\mu} K$ (say). The surviving components of $S^i{}_{jk}$ are

$$S^4{}_{23} = -S^4{}_{32} = V^4 S_{23} = e^{-(\nu+\mu)/2}. \tag{3.2}$$

With the help of (3.1)–(3.2) the set of equations (2.1) reduces to

$$-e^{-\mu}|\mu'' + \tfrac{1}{2}\mu'^2 + 2\mu'/r| = 8\pi(\rho - 2\pi K^2), \tag{3.3}$$

$$e^{-\mu}|(\nu' + \mu')/r + \tfrac{1}{2}\mu'(\nu' + \tfrac{1}{2}\mu')| = -16\pi^2 K^2, \tag{3.4}$$

$$\tfrac{1}{2}e^{-\mu}|\nu'' + \mu'' + \tfrac{1}{2}\nu'^2 + (\nu' + \mu')/r| = -16\pi^2 K^2. \tag{3.5}$$

Equations (3.3)–(3.5) are three independent equations involving four unknowns: ν, μ, ρ and K. The system is undetermined. As the space–time corresponding to the effective density $\rho - 2\pi K^2$ and the effective pressure $-16\pi^2 K^2$ has the Riemannian structure, we assume that Weyl's conformal curvature vanishes in order to have a completely determined system. The vanishing of the conformal curvature tensor implies that (Banerjee and Som 1981)

$$e^{\nu-\mu} = (A + Br^2)^2 \tag{3.6}$$

where A and B are arbitrary constants.

Equations (3.4)–(3.6) yield

$$e^{-\mu} = (a + br^2)^2 \tag{3.7}$$

where a and b are constants of integration.

The constants a, b, A and B can be obtained from boundary conditions. Since the torsion does not contribute outside the sphere, the metric is represented by the Schwarzschild vacuum solution in the isotropic coordinates

$$ds^2 = -\left(1+\frac{m}{2r}\right)^4 (dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + \frac{(1-m/2r)^2}{(1+m/2r)^2}\, dt^2 \tag{3.8}$$

where m is the Schwarzschild mass. With this we use the boundary conditions at $r = r_0$

$$g_{rr}^{\text{int}} = g_{rr}^{\text{ext}}, \qquad \frac{\partial}{\partial r} g_{rr}\Big|^{\text{int}} = \frac{\partial}{\partial r} g_{rr}\Big|^{\text{ext}}, \tag{3.9}$$

$$g_{44}^{\text{int}} = g_{44}^{\text{ext}}, \qquad \frac{\partial}{\partial r} g_{44}\Big|^{\text{int}} = \frac{\partial}{\partial r} g_{44}\Big|^{\text{ext}}. \tag{3.10}$$

From (3.9) we get

$$a = (1 + m/2r_0)^{-3}, \qquad b = m/2r_0^3 (1 + m/2r_0)^{-3}, \tag{3.11a}$$

and from (3.10) we get, using (3.11a),

$$A = \frac{1 - m/r_0}{(1 + m/2r_0)^4}, \qquad B = \frac{(m/4r_0^3)(4 - m/r_0)}{(1 + m/2r_0)^4}. \tag{3.11b}$$

Since $-16\pi^2 K^2 < 0$ we are faced with an interesting situation: one can consider a static spherical dust distribution solely in equilibrium under the influence of the spin of the constituent particles. The spin density is given by

$$16\pi^2 K^2 = -[4/(A + Br^2)]\,|a(Ba - 2Ab) + br^2(Ab - 2Ba)|. \tag{3.12}$$

For $2r_0 > m > r_0$ we have

$$4\pi^2 K^2 > 0 \qquad \text{at } r = 0 \tag{3.13}$$

and

$$4\pi^2 K^2 = 0 \qquad \text{at } r = r_0 \tag{3.14}$$

The proper density ρ of the dust distribution is given by

$$8\pi\rho = 12ab + 4a(2Ab - Ba)/A \qquad \text{at } r = 0 \tag{3.15}$$

and

$$8\pi\rho = 12ab \qquad \text{at } r = r_0. \tag{3.16}$$

The mass density as well as the spin density is maximum at the origin of the isotropic coordinates. While the spin density vanishes at $r = r_0$, the mass density assumes a finite value.

4. Discussion

We have obtained the conformally flat solutions of the Einstein–Cartan equations for a static dust sphere. The conformal solution is possible due to the fact that the Einstein–Cartan equations reduce to the Einstein equations with the modified density and pressure. The interesting feature of the solutions is that the equilibrium configuration of a pressure free fluid can exist under the influence of the spins of the constituent particles. If we compare this result with the static flat solution of a charged dust sphere, one finds that while the contribution of the electrostatic field to T_0^0 is positive, that of the classical spin density is negative. The repulsive field of the classical spin density gives rise to negative energy density. As a result, a higher density of matter can be in equilibrium for a given value of the radial coordinate under the influence of torsion and spin.

Acknowledgment

We are grateful to Dr A Banerjee for his stimulating discussion. We also acknowledge gratefully financial support from the FINEP and the CNPq of Brazil.

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